

A NOTE ON DEFORMATIONS OF REGULAR EMBEDDINGS

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ABSTRACT. In this paper we give a description of the first order deformation space of a regular embedding $X \hookrightarrow Y$ of reduced algebraic schemes. We compare our result with results of Ran (in particular [Ran, Prop. 1.3]).

This paper is dedicated to Philippe Ellia on the occasion of his sixtieth birthday.

INTRODUCTION

The deformation theory of morphisms $f : X \rightarrow Y$ between schemes over an algebraically closed field \mathbf{k} is an important subject, with a lot of applications, and there is a vast literature concerning it (e.g., see References). In general, there is a cotangent complex in a derived category of right bounded complexes giving rise to \mathbf{k} -vector spaces T_f^i , with T_f^1 the tangent space to the deformation functor of f and T_f^2 the corresponding obstruction space. When Y is smooth and f is a regular embedding, the description of T_f^1 and T_f^2 can be found in [Se, §3]. The aim of this note is to extend this description to the more general case in which Y is singular, but reduced. The main result is Proposition 1.3. The techniques we use are standard in deformation theory.

This problem has been studied also by Z. Ran in [Ran], where the author makes a proposal for a *classifying space* (Ran's terminology) for first order deformations of any morphism $f : X \rightarrow Y$. In §1.4 we compare Ran's result with ours (in the regular embedding case) and we observe that Ran's space maps to T_f^1 , but, in general, the map is not an isomorphism.

The reason we got involved in this topic, has been our work on the universal Severi variety of nodal curves on the moduli spaces of polarised $K3$ surfaces and on the related moduli map, cf. [CFGK1, CFGK2]. In §2 we make some considerations on this subject and explain how our results can be used to attack the study of moduli problems for Severi varieties by degeneration arguments (as we did in [CFGK2]). We think that these ideas can be usefully applied to investigate still unexplored areas like moduli problems for Severi varieties on surfaces other than $K3$ s, e.g., Enriques surfaces.

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Terminology. For deformation theory we refer to [Se, Hart2, GLS]. We will use the same notation and terminology as in [Se]. In particular, a closed embedding $\nu : X \hookrightarrow Y$ of algebraic schemes will mean a closed immersion as in [Hart, p.85]. For a closed embedding $\nu : X \hookrightarrow Y$ we will set $\Omega_Y^1|_X := \nu^*(\Omega_Y^1)$.

1. FIRST ORDER DEFORMATIONS OF CLOSED REGULAR EMBEDDINGS

1.1. Preliminaries. Let X and Y be reduced (noetherian and separated) algebraic schemes over an algebraically closed field \mathbf{k} and let $\nu : X \hookrightarrow Y$ be a *regular closed embedding* of codimension r , i.e., X , as a subscheme of Y , is locally defined by a regular sequence (cf. [Se, App. D]). We want to study first order deformations of ν without assuming, as in [Se, §3.4.4], that Y is smooth.

A first order deformation $\tilde{\nu} : \mathcal{X} \rightarrow \mathcal{Y}$ of ν is a cartesian diagram

$$\begin{array}{ccc} X & \xhookrightarrow{i} & \mathcal{X} \\ \downarrow \nu & & \downarrow \tilde{\nu} \\ Y & \xhookrightarrow{j} & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \xhookrightarrow{s} & \mathrm{Spec}(\mathbf{k}[\epsilon]) \end{array}$$

where \mathcal{X} and \mathcal{Y} are flat over $\mathrm{Spec}(\mathbf{k}[\epsilon])$ (cf. [Se, Def. 3.4.1]). In particular \mathcal{X} and \mathcal{Y} are infinitesimal deformations of X and Y respectively. The notions of isomorphic, trivial and locally trivial deformations of \mathcal{X} , \mathcal{Y} and ν are given as usual.

Remark 1.1. (1) In the diagram above ν is a closed embedding (cf. [Se, Note 3, p. 185]).

(2) If $\tilde{\nu} : \mathcal{X} \hookrightarrow \mathcal{Y}$ is a first order deformation of ν and \mathcal{Y} is trivial, then $\tilde{\nu}$ is a regular embedding. This follows by flatness using [Se, Cor. A.11].

(3) If Y is a Cohen-Macaulay scheme and $\tilde{\nu} : \mathcal{X} \hookrightarrow \mathcal{Y}$ is a first order deformation of ν , then $\tilde{\nu}$ is again a regular embedding. Indeed, since Y is Cohen-Macaulay, then \mathcal{Y} is Cohen-Macaulay (cf. [Ma, Thm. 24.5] or [Stack, §10.129]) and the result follows by [Hart2, Thm. 9.2].

From now on we will denote by Def_ν and Def_Y (resp. Def'_ν and Def'_Y) the functors of deformations (resp. first order locally trivial deformations) of ν and Y respectively, whose properties are described in the aforementioned references, especially in [Se]. For the functors Def_ν and Def_Y there is a cotangent complex in a derived category of right bounded complexes giving rise to the \mathbf{k} -vector spaces T_ν^i and T_Y^i . Then T_ν^1 and T_Y^1 are the tangent spaces to Def_ν and Def_Y and T_ν^2 and T_Y^2 the corresponding obstruction spaces (cf. [GLS, Thm. C.5.1, Cor. C.5.2] and related references as [LiS, Flen, Fle1, Illu1, Illu2, Buc]). The aim of this section is to describe the vector space $T_\nu^1 \simeq \mathrm{Def}_\nu(\mathbf{k}[\epsilon])$.

Let \mathcal{I} be a coherent locally free sheaf on Y . We will use the standard identifications of vector spaces

$$(1) \quad \mathrm{Ex}_{\mathbf{k}}(Y, \mathcal{I}) \simeq \mathrm{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1, \mathcal{I}),$$

where $\mathrm{Ex}_{\mathbf{k}}(Y, \mathcal{I}) := \mathrm{Ex}(Y/\mathrm{Spec}(\mathbf{k}), \mathcal{I})$ is the space of (infinitesimal) extensions of Y by \mathcal{I} , and

$$(2) \quad \mathrm{Def}_Y(\mathbf{k}[\epsilon]) \simeq \mathrm{Ex}_{\mathbf{k}}(Y, \mathcal{O}_Y) \simeq \mathrm{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1, \mathcal{O}_Y) \quad \text{and} \quad \mathrm{Def}'_Y(\mathbf{k}[\epsilon]) \simeq H^1(Y, \Theta_Y),$$

where $\Theta_Y = \mathcal{H}om_{\mathcal{O}_Y}(\Omega_Y^1, \mathcal{O}_Y)$, while $H^0(Y, \Theta_Y)$ is the space of infinitesimal automorphisms of Y (cf. [Se, Thms. 1.1.10 and 2.4.1 and Prop. 2.6.2]).

Remark 1.2. The deformation theory of a regular closed embedding is easier than the one of an arbitrary closed embedding because of the properties of its conormal sequence. Indeed, if $\nu : X \hookrightarrow Y$ is a regular closed embedding of codimension r and I is the ideal sheaf of $X \subseteq Y$, then I/I^2 is a locally free sheaf on X of rank r . This implies that the conormal sequence

$$(3) \quad 0 \longrightarrow I/I^2 \longrightarrow \Omega_Y^1|_X \xrightarrow{\beta} \Omega_X^1 \longrightarrow 0$$

of ν is exact on the left.

1.2. The description of the first order deformation space. Recall the natural morphisms

$$\mu : \mathrm{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1, \mathcal{O}_Y) \longrightarrow \mathrm{Ext}_{\mathcal{O}_X}^1(\Omega_Y^1|_X, \mathcal{O}_X) \quad \text{and} \quad \lambda : \mathrm{Ext}_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) \longrightarrow \mathrm{Ext}_{\mathcal{O}_X}^1(\Omega_Y^1|_X, \mathcal{O}_X),$$

where:

- μ restricts an extension $0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{E} \rightarrow \Omega_Y^1 \rightarrow 0$ to X ;
- λ sends an extension $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow \Omega_X^1 \rightarrow 0$ to the fiber product $\mathcal{E} \times_{\Omega_X^1} \Omega_Y^1|_X$ via β in (3).

Proposition 1.3. *Let $\nu : X \hookrightarrow Y$ be a regular closed embedding of reduced algebraic schemes. Then the first order deformation space of ν is isomorphic to the fiber product*

$$(4) \quad \begin{array}{ccc} \mathrm{Def}_\nu(\mathbf{k}[\epsilon]) \simeq \mathrm{Ext}_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) \times_{\mathrm{Ext}_{\mathcal{O}_X}^1(\Omega_Y^1|_X, \mathcal{O}_X)} \mathrm{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1, \mathcal{O}_Y) & \xrightarrow{p_Y} & \mathrm{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1, \mathcal{O}_Y) \\ p_X \downarrow & & \downarrow \mu \\ \mathrm{Ext}_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) & \xrightarrow{\lambda} & \mathrm{Ext}_{\mathcal{O}_X}^1(\Omega_Y^1|_X, \mathcal{O}_X). \end{array}$$

Proof. Let $s \in \mathrm{Ext}_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X)$ and $t \in \mathrm{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1, \mathcal{O}_Y)$ correspond to first order deformations $\mathcal{X} \rightarrow \mathrm{Spec}(\mathbf{k}[\epsilon])$ and $\mathcal{Y} \rightarrow \mathrm{Spec}(\mathbf{k}[\epsilon])$ of X and Y respectively, with closed embeddings $i : X \hookrightarrow \mathcal{X}$ and $j : Y \hookrightarrow \mathcal{Y}$, cf. (2). With this interpretation, we have

$$s := (0 \rightarrow \mathcal{O}_Y \rightarrow \Omega_Y^1|_Y \rightarrow \Omega_Y^1 \rightarrow 0) \xrightarrow{\mu} \mu(s) := (0 \rightarrow \mathcal{O}_X \rightarrow \Omega_Y^1|_X \rightarrow \Omega_Y^1|_X \rightarrow 0)$$

and

$$t := (0 \rightarrow \mathcal{O}_X \rightarrow \Omega_X^1|_X \rightarrow \Omega_X^1 \rightarrow 0) \xrightarrow{\lambda} \lambda(t) := (0 \rightarrow \mathcal{O}_X \rightarrow \Omega_X^1|_X \times_{\Omega_X^1} \Omega_Y^1|_X \rightarrow \Omega_Y^1|_X),$$

so that we have a commutative diagram

$$(5) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & 0 & \longrightarrow & \mathcal{O}_X & \xlongequal{\quad} & \mathcal{O}_X & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & I/I^2 & \longrightarrow & \Omega_Y^1|_X \times_{\Omega_X^1} \Omega_X^1|_X & \xrightarrow{L} & \Omega_X^1|_X \longrightarrow 0 \\ & \parallel & & \downarrow \alpha & & \downarrow M & \\ 0 & \longrightarrow & I/I^2 & \longrightarrow & \Omega_Y^1|_X & \xrightarrow{\beta} & \Omega_X^1|_X \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

where I is the ideal sheaf of X in Y , as in Remark 1.2.

We also observe that $j\nu : X \hookrightarrow Y \hookrightarrow \mathcal{Y}$ is a regular closed embedding and \mathcal{Y} is reduced. The conormal exact sequences (see Remark 1.2) of ν , j and $j\nu$ fit in the commutative diagram

$$(6) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{I}/\mathcal{I}^2 & \longrightarrow & I/I^2 \longrightarrow 0 \\ & \parallel & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \Omega_Y^1|_X & \longrightarrow & \Omega_Y^1|_X \longrightarrow 0 \\ & & & \downarrow \gamma & & \downarrow & \\ & & & \Omega_X^1 & \xlongequal{\quad} & \Omega_X^1 & \\ & & & \downarrow & & \downarrow & \\ & & & 0 & & 0 & \end{array}$$

where \mathcal{I} is the ideal sheaf of X in \mathcal{Y} .

To prove the proposition, we want to prove that $\lambda(s) = \mu(t)$ if and only if there exists a closed embedding

$$(7) \quad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\tilde{\nu}} & \mathcal{Y} \\ & \searrow & \downarrow \\ & & \mathrm{Spec}(\mathbf{k}[\epsilon]) \end{array}$$

restricting to $\nu : X \rightarrow Y$ over $\mathrm{Spec}(\mathbf{k})$, i.e. such that $\tilde{\nu}i = j\nu$.

First assume that $\lambda(s) = \mu(t)$, i.e., there exists a commutative diagram

$$(8) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \Omega_{\mathcal{Y}|X}^1 & \longrightarrow & \Omega_{\mathcal{Y}|X}^1 \longrightarrow 0 \\ & & \parallel & & \downarrow \simeq & & \parallel \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \Omega_Y^1|_X \times_{\Omega_X^1} \Omega_X^1|_X & \longrightarrow & \Omega_Y^1|_X \longrightarrow 0. \end{array}$$

By (8), the map $\beta\alpha = ML$ in (5) can be identified with the map γ in (6), whose kernel is

$$\mathcal{I}/\mathcal{I}^2 \simeq \ker(\beta\alpha) = \ker(ML) \simeq I/I^2 \oplus \mathcal{O}_X.$$

In particular the conormal sequence of X in \mathcal{Y} , which is the central vertical sequence in (6), fits in the commutative diagram

$$(9) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I/I^2 & \longrightarrow & I/I^2 \oplus \mathcal{O}_X & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I/I^2 & \longrightarrow & \Omega_{\mathcal{Y}|X}^1 & \xrightarrow{L} & \Omega_X^1|_X \longrightarrow 0 \\ & & \downarrow & & \downarrow \beta\alpha=\gamma & & \downarrow M \\ & & 0 & \longrightarrow & \Omega_X^1 & \xlongequal{\quad} & \Omega_X^1 \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Since $I/I^2 \oplus \mathcal{O}_X$ is locally free, by (9) we obtain the new diagram

$$(10) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I/I^2 & \longrightarrow & I/I^2 \oplus \mathcal{O}_X & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I/I^2 & \longrightarrow & \mathcal{O}_{\mathcal{Y}} & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & \mathcal{O}_X & \xlongequal{\quad} & \mathcal{O}_X \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

where the map $\mathcal{O}_{\mathcal{Y}} \rightarrow \mathcal{O}_{\mathcal{X}}$ (which provides the desired embedding $\mathcal{X} \subseteq \mathcal{Y}$) is induced by L in (9), and by the isomorphisms (see [Se, Proof of Thm. 1.1.10])

$$\mathcal{O}_{\mathcal{Y}} \simeq \Omega_{\mathcal{Y}|X}^1 \times_{\Omega_X^1} \mathcal{O}_X \quad \text{and} \quad \mathcal{O}_{\mathcal{X}} \simeq \Omega_{\mathcal{X}|X}^1 \times_{\Omega_X^1} \mathcal{O}_X$$

where the fiber products are between the derivation $d : \mathcal{O}_X \rightarrow \Omega_X^1$ and the conormal maps.

Conversely, assume that there is a closed embedding (7) such that $\tilde{\nu}i = j\nu$. Then one obtains a diagram like (5), with $\Omega_Y^1|_X \times_{\Omega_X^1} \Omega_{\mathcal{X}}^1|_X$ replaced by $\Omega_{\mathcal{Y}|X}^1$. Using the universal property of the fiber product, one deduces an isomorphism of extensions as in (8), ending the proof of the proposition. \square

Remark 1.4. Suitable versions of the maps λ and μ can be defined even if the embedding ν is not regular. However our proof of Proposition 1.3 does not extend, as it is, to this more general case, because the kernel of the conormal sequence of X in Y is no longer locally free.

1.3. Comments. To better understand the maps λ and μ , and the related geometry, observe that they fit in the following diagram with exact rows and columns:

$$(11) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \text{Hom}_{\mathcal{O}_Y}(\Omega_Y^1, I) & = & \text{Hom}_{\mathcal{O}_Y}(\Omega_Y^1, I) & & \\ & & \downarrow H^0(\Theta_Y) & = & \downarrow H^0(\Theta_Y) & & \\ & & \text{Hom}_{\mathcal{O}_X}(\Omega_Y^1|_X, \mathcal{O}_X) & = & \text{Hom}_{\mathcal{O}_X}(\Omega_Y^1|_X, \mathcal{O}_X) & & \\ & & \downarrow \text{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1, I) & = & \downarrow \text{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1, I) & & \\ 0 \rightarrow H^0(\Theta_X) \rightarrow \text{Hom}_{\mathcal{O}_X}(\Omega_Y^1|_X, \mathcal{O}_X) \rightarrow H^0(N_{X/Y}) \rightarrow & \text{Def}_{\nu}(\mathbf{k}[\epsilon]) & \xrightarrow{p_Y} & \text{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1, \mathcal{O}_Y) & \rightarrow & H^1(N_{X/Y}) & \\ & \downarrow p_X & & \downarrow \mu & & \parallel & \\ 0 \rightarrow H^0(\Theta_X) \rightarrow \text{Hom}_{\mathcal{O}_X}(\Omega_Y^1|_X, \mathcal{O}_X) \rightarrow H^0(N_{X/Y}) \rightarrow & \text{Ext}_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) & \xrightarrow{\lambda} & \text{Ext}_{\mathcal{O}_X}^1(\Omega_Y^1|_X, \mathcal{O}_X) & \rightarrow & H^1(N_{X/Y}) & \\ & \downarrow & & \downarrow & & \downarrow & \\ & \text{Ext}_{\mathcal{O}_Y}^2(\Omega_Y^1, I) & = & \text{Ext}_{\mathcal{O}_Y}^2(\Omega_Y^1, I) & & \text{Ext}_{\mathcal{O}_X}^2(\Omega_X^1, \mathcal{O}_X) & \\ & \downarrow & = & \downarrow & & \downarrow & \\ & \text{Ext}_{\mathcal{O}_Y}^2(\Omega_Y^1, \mathcal{O}_Y) & = & \text{Ext}_{\mathcal{O}_Y}^2(\Omega_Y^1, \mathcal{O}_Y) & & \text{Ext}_{\mathcal{O}_X}^2(\Omega_Y^1|_X, \mathcal{O}_X) & \\ & \downarrow & & \downarrow & & \downarrow & \\ & \text{Ext}_{\mathcal{O}_Y}^2(\Omega_Y^1, \mathcal{O}_X) & = & \text{Ext}_{\mathcal{O}_Y}^2(\Omega_Y^1, \mathcal{O}_X) & & \vdots & \\ & \vdots & & \vdots & & \vdots & \end{array}$$

where the lower row arises from the conormal sequence of $\nu : X \hookrightarrow Y$ and the second column from $0 \rightarrow I \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_X \rightarrow 0$. We denoted by $N_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X)$ the normal sheaf of X in Y , we used the standard isomorphisms $H^i(X, N_{X/Y}) \simeq \text{Ext}_{\mathcal{O}_X}^i(I/I^2, \mathcal{O}_X)$ (using that I/I^2 is locally free) and one checks that $\text{Ext}_{\mathcal{O}_X}^i(\Omega_Y^1|_X, \mathcal{O}_X) \simeq \text{Ext}_{\mathcal{O}_Y}^i(\Omega_Y^1, \mathcal{O}_X)$, for $i = 0, 1$. The space $H^0(X, N_{X/Y})$ consists of first order deformations of X as a subscheme of (the fixed scheme) Y , while $H^1(X, N_{X/Y})$ is the corresponding obstruction space (because X is regularly embedded in Y), and the map $H^0(N_{X/Y}) \rightarrow \text{Def}_{\nu}(\mathbf{k}[\epsilon])$ is the obvious morphism.

The diagram (11) and the cotangent braid [GLS, p. 446] suggest that $\text{Hom}_{\mathcal{O}_X}(\Omega_Y^1|_X, \mathcal{O}_X)$ should be the first order deformation space of ν preserving X and Y (cf. [Se, §3.4.1]), while $\text{Ext}_{\mathcal{O}_X}^1(\Omega_Y^1|_X, \mathcal{O}_X)$ should be the corresponding obstruction space. This is the case if Y is smooth (see [Se, Prop. 3.4.2]). The first fact follows from the following more general result.

Proposition 1.5. *Let $f : X \rightarrow Y$ be a morphism of reduced algebraic schemes and let $\text{Def}_{X/f/Y}$ be the deformation functor of f preserving X and Y . Then*

$$(12) \quad \text{Def}_{X/f/Y}(\mathbf{k}(\epsilon)) \simeq \text{Hom}_{\mathcal{O}_X}(f^*\Omega_Y^1, \mathcal{O}_X).$$

If $\text{Hom}_{\mathcal{O}_X}(f^\Omega_Y^1, \mathcal{O}_X)$ has finite dimension (in particular if X is projective), then $\text{Def}_{X/f/Y}$ is pro-representable.*

Proof. Let $j : \Gamma \hookrightarrow X \times Y$ be the embedding of the graph of f in $X \times Y$ and $q : X \times Y \rightarrow Y$ and $p : X \times Y \rightarrow X$ the natural projections. Then, by arguing as in step (i) of the proof of [Se, Prop. 3.4.2], we find a natural isomorphism of functors between $\text{Def}_{X/f/Y}$ and the local Hilbert functor $H_\Gamma^{X \times Y}$. In particular, by [Se, Prop. 3.2.1], we have

$$\text{Def}_{X/\nu/Y}(\mathbf{k}(\epsilon)) \simeq H_\Gamma^{X \times Y}(\mathbf{k}(\epsilon)) \simeq H^0(\Gamma, N_{\Gamma|X \times Y}) \simeq \text{Hom}_{\mathcal{O}_\Gamma}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_\Gamma),$$

where \mathcal{J} is the ideal sheaf of Γ in $X \times Y$. Now observe that Γ and X are isomorphic via pj . Hence Γ is reduced and the conormal sequence of Γ in $X \times Y$ can be written as

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{X \times Y}^1|_\Gamma \longrightarrow \Omega_\Gamma^1 \longrightarrow 0,$$

where \mathcal{A} is a torsion sheaf on Γ . Moreover we have the (split) exact sequence

$$0 \longrightarrow q^* \Omega_Y^1 \longrightarrow \Omega_{X \times Y}^1 \longrightarrow p^* \Omega_X^1 \longrightarrow 0.$$

By restricting this to Γ one finds the exact sequence

$$0 \longrightarrow j^* q^* \Omega_Y^1 \longrightarrow j^* \Omega_{X \times Y}^1 = \Omega_{X \times Y}^1|_\Gamma \longrightarrow j^* p^* \Omega_X^1 \longrightarrow 0.$$

Since $j^* p^* \Omega_X^1$ and Ω_Γ^1 are isomorphic, one obtains an isomorphism $j^* q^* \Omega_Y^1 \simeq (\mathcal{J}/\mathcal{J}^2)/\mathcal{A}$. Moreover, $j^* q^* \Omega_Y^1 = (pj)^* f^* \Omega_Y^1$. It follows that

$$\text{Def}_{X/\nu/Y}(\mathbf{k}(\epsilon)) \simeq \text{Hom}_{\mathcal{O}_\Gamma}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_\Gamma) \simeq \text{Hom}_{\mathcal{O}_\Gamma}((\mathcal{J}/\mathcal{J}^2)/\mathcal{A}, \mathcal{O}_\Gamma) \simeq \text{Hom}_{\mathcal{O}_X}(f^* \Omega_Y^1, \mathcal{O}_X).$$

The last statement of the proposition follows by the isomorphism of functors $H_\Gamma^{X \times Y} \simeq \text{Def}_{X/f/Y}$ and the fact that, under the hypothesis, $H_\Gamma^{X \times Y}$ is pro-representable (cf. [Se, Cor. 3.2.2]). \square

Remark 1.6. The proof of Proposition 1.5 adapts the proof of [Se, Prop. 3.4.2] to the singular case. Note that, if Y is singular, the graph of $f : X \rightarrow Y$ is in general no longer a regular embedding, even if f is a regular embedding. Therefore

$$H^1(\Gamma, N_{\Gamma|X \times Y}) \simeq H^1(\text{Hom}_{\mathcal{O}_X}(f^* \Omega_Y^1, \mathcal{O}_X)) \subseteq \text{Ext}_{\mathcal{O}_X}^1(f^* \Omega_Y^1, \mathcal{O}_X)$$

is not necessarily the obstruction space for $H_\Gamma^{X \times Y} \simeq \text{Def}_{X/f/Y}$.

1.4. Comparison with results by Z. Ran. In [Ran] there is a proposal for a *classifying space* for first order deformations of a morphism $f : X \rightarrow Y$, with X, Y any pair of schemes. Given $f : X \rightarrow Y$ there are two obvious maps

$$\delta_0 : f^* \mathcal{O}_Y \longrightarrow \mathcal{O}_X \quad \text{and} \quad \delta_1 : f^* \Omega_Y^1 \longrightarrow \Omega_X^1.$$

In [Ran] one constructs vector spaces $\text{Ext}^i(\delta_1, \delta_0)$, $i \geq 0$, fitting in the long exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Hom}(\delta_1, \delta_0) \longrightarrow \text{Hom}_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X) \oplus \text{Hom}_{\mathcal{O}_Y}(\Omega_Y^1, \mathcal{O}_Y) \xrightarrow{\varphi_0} \text{Hom}_{\mathcal{O}_X}(f^* \Omega_Y^1, \mathcal{O}_X) \xrightarrow{\partial} \\ \xrightarrow{\partial} \text{Ext}^1(\delta_1, \delta_0) \longrightarrow \text{Ext}_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) \oplus \text{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1, \mathcal{O}_Y) \xrightarrow{\varphi_1} \text{Ext}_{\mathcal{O}_X}^1(f^* \Omega_Y^1, \mathcal{O}_X) \longrightarrow \dots \end{aligned}$$

where

$$\varphi_0(\alpha, \beta) = \alpha \delta_1 - \delta_0 f^* \beta$$

and the map φ_1 is defined accordingly. The result in [Ran] is that $\text{Ext}^1(\delta_1, \delta_0)$ is the classifying space in question (see [Ran, Prop. 3.1]).

We remark that $\text{Ext}^1(\delta_1, \delta_0)$ does not, in general, coincide with $\text{Def}_f(\mathbf{k}[\epsilon])$. Indeed, let us consider the case in which $f : X \rightarrow Y$ is a regular embedding. Then, by (11), one has $\varphi_1 = \lambda - \mu$. Therefore

$$\text{Def}_f(\mathbf{k}[\epsilon]) \simeq \text{Ker}(\varphi_1).$$

We now provide an example where the map ∂ is non-zero, which shows that $\text{Ext}^1(\delta_0, \delta_1)$ surjects onto $\text{Def}_f(\mathbf{k}[\epsilon])$ but is not isomorphic to it.

Example 1.7. Let $\pi : Y \rightarrow Z$ be a smooth surjective morphism with Z smooth of positive dimension, Y irreducible such that $h^0(Y, \Theta_Y) = 0$ and irreducible fibres all isomorphic to a fixed X such that $h^0(X, \Theta_X) = 0$. Then $N_{X/Y} \simeq \mathcal{O}_X^{\dim(Z)}$ and the coboundary map $H^0(X, N_{X/Y}) \rightarrow H^1(X, \Theta_X)$ is zero, hence $h^0(X, \Theta_{Y|X}) = \dim(Z) > 0$ and ∂ is non-zero.

This situation is easy to cook up: it suffices to take $Y = X \times Z$ and $h^0(X, \Theta_X) = h^0(Z, \Theta_Z) = 0$.

2. REMARKS ON DEFORMATIONS OF NODAL CURVES ON NORMAL CROSSING SURFACES

Let S be a connected surface with (at most) normal crossing singularities and $i : C \hookrightarrow S$ be the regular embedding of a (reduced) nodal curve. Let $N \subset S$ be the length- δ scheme of nodes of C lying on the smooth locus of S and $\pi : Y \rightarrow S$ be the blowing-up at N . Denote by X the proper transform of C in Y . Then $\phi = \pi|_X : X \rightarrow S$ is the partial normalization of $C = \phi(X)$ at the nodes on the smooth locus of S . We will assume that X is connected. We will set $g := p_a(X)$ (since X is connected, then $g \geq 0$). Denote by $\nu : X \hookrightarrow Y$ the embedding of X in Y and by $\text{Def}_\phi(\mathbf{k}[\epsilon])$ the first order deformation space of ϕ .

Lemma 2.1. *There exists a natural isomorphism*

$$\text{Def}_\nu(\mathbf{k}[\epsilon]) \simeq \text{Def}_\phi(\mathbf{k}[\epsilon]).$$

Proof. One has an exact sequence

$$(13) \quad 0 \longrightarrow \mathbf{k}^{2\delta} \longrightarrow \text{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1, \mathcal{O}_Y) \longrightarrow \text{Ext}_{\mathcal{O}_S}^1(\Omega_S^1, \mathcal{O}_S) \longrightarrow 0$$

inducing a morphism $\text{Def}_\nu(\mathbf{k}[\epsilon]) \rightarrow \text{Def}_\phi(\mathbf{k}[\epsilon])$, which is an isomorphism. \square

Now assume H is a polarization on S and assume that there is an irreducible component \mathcal{B} of a moduli scheme parametrizing isomorphism classes of polarized surfaces with no worse singularities than normal crossings, such that $(S, H) \in \mathcal{B}$ and the general point of \mathcal{B} corresponds to a pair (S', H') , with S' smooth and irreducible.

One sees that there exists (at least locally) a scheme $\mathcal{V}_{m,\delta}$, called the (m, δ) -universal Severi variety, endowed with a morphism

$$\phi_{m,\delta} : \mathcal{V}_{m,\delta} \longrightarrow \mathcal{B}.$$

The points in $\mathcal{V}_{m,\delta}$ are pairs (S', C') with $(S', H') \in \mathcal{B}$, and $C' \in |mH'|$ nodal, with exactly δ nodes on the smooth locus of S' , and connected normalization at these δ nodes (in [CFGK1, § 2] we treated the special case of $K3$ surfaces).

Let \mathcal{V} be an irreducible component of $\mathcal{V}_{m,\delta}$ and assume that for $(S', C') \in \mathcal{V}$ general, the normalization of C' at the δ nodes is stable. Then one has the obvious *moduli map*

$$\psi_{m,\delta} : \mathcal{V} \dashrightarrow \overline{\mathcal{M}}_g,$$

where $\overline{\mathcal{M}}_g$ is the Deligne–Mumford compactification of the moduli space of smooth, genus g curves. Given $(S, C) \in \mathcal{V}$, if the normalization X of C at the δ nodes on the smooth locus of S is stable, then $\psi_{m,\delta}$ is defined at (S, C) . In particular, from Lemma 2.1 and diagram (11), we obtain:

Corollary 2.2. *There are natural identifications*

$$\text{T}_{(S,C)}\mathcal{V}_{m,\delta} \simeq \text{Def}_\phi(\mathbf{k}[\epsilon]) \simeq \text{Def}_\nu(\mathbf{k}[\epsilon]).$$

Moreover, if X is stable, then the map

$$p_X : \text{Def}_\nu(\mathbf{k}[\epsilon]) \simeq \text{T}_{(S,C)}\mathcal{V}_{m,\delta} \longrightarrow \text{Ext}_{\mathcal{O}_X}^1(\Omega_X, \mathcal{O}_X) \simeq \text{T}_{[X]}\overline{\mathcal{M}}_g$$

in (15) (and (11)) is the differential $d_{(S,C)}\psi_{m,\delta}$ of $\psi_{m,\delta}$ at (S, C) . In particular, if $\text{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1(X), \mathcal{O}_Y) = 0$ (resp. $\text{Ext}_{\mathcal{O}_Y}^2(\Omega_Y^1(X), \mathcal{O}_Y) = 0$), then $d_{(S,C)}\psi_{m,\delta}$ is injective (resp. surjective).

Remark 2.3. When S is a smooth surface, the tangent space to $\mathcal{V}_{m,\delta}$ at (S, C) coincides with the space of first order locally trivial deformations of $i : C \hookrightarrow S$. In particular, by [Se, Prop. 3.4.17] we know that $T_{(S,C)}\mathcal{V}_{m,\delta} \simeq H^1(S, T_S\langle C \rangle)$, where $T_S\langle C \rangle$ is the sheaf defined in [Se, (3.56)], and $\text{Ext}_{\mathcal{O}_S}^i(\Omega_S(C), \mathcal{O}_S) \simeq H^i(S, T_S(-C))$, for $i = 0, 1, 2$, as observed in [FKPS] in the case S is a $K3$ surface.

In general the computation of the cohomology groups appearing in diagram (11), hence in the statement of Lemma 2, is difficult. A possible approach to this problem is by degeneration, as we explain now.

Since the surface Y has normal crossing, there is on Y a locally free sheaf Λ_Y^1 (cf. [Fr, Thm. (3.2)]) encoding important information about deformations of Y . If $f : \mathcal{Y} \rightarrow \Delta$ is any semi-stable deformation of $Y = f^{-1}(0)$ [Fr, (1.12)] with general fibre Y_t , there is a locally free sheaf $\Omega_{\mathcal{Y}/\Delta}(\log Y)$ on \mathcal{Y} such that $\Omega_{\mathcal{Y}/\Delta}(\log Y)|_{Y_t} \simeq \Omega_{Y_t}^1$ and

$$(14) \quad \Lambda_Y^1 \simeq \Omega_{\mathcal{Y}/\Delta}(\log Y)|_Y$$

(see [Fr, §3], [CFGK2, §2] and related references).

Let $\mathcal{X} \subset \mathcal{Y} \rightarrow \Delta$ be a flat family of curves with fibres X_t , for $t \in \Delta$ and $X_0 = X$ as above. Then, by flatness and semicontinuity, one has

$$\dim(\text{Ext}_{\mathcal{O}_Y}^i(\Lambda_Y^1 \otimes \mathcal{O}_Y(X), \mathcal{O}_Y)) \geq \dim(\text{Ext}_{\mathcal{O}_{Y_t}}^i(\Omega_{Y_t}^1 \otimes \mathcal{O}_{Y_t}(X_t), \mathcal{O}_{Y_t})), \quad \text{for } t \in \Delta \text{ general.}$$

Hence, in order to prove that the moduli map is generically of maximal rank, it suffices to prove vanishing theorems for $\text{Ext}_{\mathcal{O}_Y}^i(\Lambda_Y^1(X), \mathcal{O}_Y)$, for $i = 1$ or $i = 2$, on Y , which in certain cases may be easier to obtain than the vanishings of $\text{Ext}_{\mathcal{O}_{Y_t}}^i(\Omega_{Y_t}^1 \otimes \mathcal{O}_{Y_t}(X_t), \mathcal{O}_{Y_t})$ on the general Y_t .

This approach has proved to be useful in the case of $K3$ surfaces (see [CFGK2]).

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ERRATA CORRIGE TO
“A NOTE ON DEFORMATIONS OF REGULAR EMBEDDINGS”

The main result of the paper [CFGK] (Proposition 1.3) is wrongly stated. Nevertheless the proof of Proposition 1.3 and Proposition 1.5 provide a complete description of $\text{Def}_\nu(\mathbf{k}[\epsilon])$ and the paper needs only the corrections below.

CORRECTIONS

• The statement of Proposition 1.3 has to be replaced by the following, which is exactly what is proved.

Proposition 1.3. Let $\nu : X \hookrightarrow Y$ be a regular closed embedding of reduced algebraic schemes and let $\text{Def}_{X/\nu/Y}$ be the deformation functor of ν preserving X and Y (cf. [Se, §3.4.1]). Then there exists a surjective morphism Φ from $\text{Def}_\nu(\mathbf{k}[\epsilon])$ to the fiber product

$$(4) \quad \begin{array}{ccc} \text{Def}_\nu(\mathbf{k}[\epsilon]) & & \\ \Phi \downarrow & & \\ \text{Ext}_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) \times_{\text{Ext}_{\mathcal{O}_X}^1(\Omega_Y^1|_X, \mathcal{O}_X)} \text{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1, \mathcal{O}_Y) & \xrightarrow{p_Y} & \text{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1, \mathcal{O}_Y) \\ p_X \downarrow & & \downarrow \mu \\ \text{Ext}_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) & \xrightarrow{\lambda} & \text{Ext}_{\mathcal{O}_X}^1(\Omega_Y^1|_X, \mathcal{O}_X) \end{array}$$

whose kernel is the image of the natural map $\Delta : \text{Def}_{X/\nu/Y}(\mathbf{k}[\epsilon]) \longrightarrow \text{Def}_\nu(\mathbf{k}[\epsilon])$.

Recalling that

$$(\dagger) \quad \text{Def}_{X/\nu/Y}(\mathbf{k}[\epsilon]) \simeq \text{Hom}_{\mathcal{O}_X}(\nu^*\Omega_Y^1, \mathcal{O}_X) = \text{Hom}_{\mathcal{O}_X}(\Omega_Y^1|_X, \mathcal{O}_X),$$

by Proposition 1.5, we obtain the following result describing $\text{Def}_\nu(\mathbf{k}[\epsilon])$, which is now to be considered the main result of the paper. (In the statement, the map $\beta : \Omega_Y^1|_X \longrightarrow \Omega_X^1$ is the one in the conormal sequence.)

Theorem. Let $\nu : X \hookrightarrow Y$ be a regular closed embedding of reduced algebraic schemes. Then there exists a long exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Hom}_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X) \times_{\text{Hom}_{\mathcal{O}_X}(\Omega_Y^1|_X, \mathcal{O}_X)} \text{Hom}_{\mathcal{O}_Y}(\Omega_Y^1, \mathcal{O}_Y) &\longrightarrow \text{Hom}_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X) \times \text{Hom}_{\mathcal{O}_Y}(\Omega_Y^1, \mathcal{O}_Y) \xrightarrow{\Theta} \\ &\xrightarrow{\Theta} \text{Hom}_{\mathcal{O}_X}(\Omega_Y^1|_X, \mathcal{O}_X) \xrightarrow{\Delta} \text{Def}_\nu(\mathbf{k}[\epsilon]) \xrightarrow{\Phi} \text{Ext}_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) \times_{\text{Ext}_{\mathcal{O}_X}^1(\Omega_Y^1|_X, \mathcal{O}_X)} \text{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1, \mathcal{O}_Y) \longrightarrow 0 \end{aligned}$$

where the map Θ is given by $\Theta(\xi, \eta) = \xi \circ \beta - \eta|_X$.

Proof. The second row of the above exact sequence follows from (the above version of) Proposition 1.3 and (\dagger) .

By the definition of $\text{Hom}_{\mathcal{O}_X}(\nu^*\Omega_Y^1, \mathcal{O}_X)$ and $\text{Def}_\nu(\mathbf{k}[\epsilon])$ (cf. [Se, p. 158 and p. 177]), an element mapped to zero by Δ corresponds to a first order deformation

$$\tilde{\nu} : X \times \text{Spec}(\mathbf{k}[\epsilon]) \rightarrow Y \times \text{Spec}(\mathbf{k}[\epsilon])$$

of ν that is trivializable. More precisely, denoting by $H_X \subset \text{Aut}(X \times \text{Spec}(\mathbf{k}[\epsilon]))$ the space of automorphisms restricting to the identity on the closed fibre and similarly for $H_Y \subset \text{Aut}(Y \times \text{Spec}(\mathbf{k}[\epsilon]))$, there exist $\alpha \in H_X$ and $\beta \in H_Y$, such that

$$\alpha \circ (\nu \times \text{id}_{\text{Spec}(\mathbf{k}[\epsilon])}) \circ \beta = \tilde{\nu}.$$

Then one obtains a natural map $H_X \times H_Y \rightarrow \text{Def}_{X/\nu/Y}(\mathbf{k}[\epsilon])$ whose image is the kernel of Δ . By (\dagger) and the well-known isomorphisms $H_X \simeq \text{Hom}_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$ and $H_Y \simeq \text{Hom}_{\mathcal{O}_Y}(\Omega_Y^1, \mathcal{O}_Y)$ (cf. [Se, Lemma 1.2.6]), this map may be identified with Θ . The kernel of Θ is by definition as in the statement. \square

- In the first column of diagram (11) the vector space $\text{Def}_\nu(\mathbf{k}[\epsilon])$ must be replaced by the quotient $\text{Def}_\nu(\mathbf{k}[\epsilon])/\text{Im}(\Delta)$.

- The paragraph “We remark that $\text{Ext}^1(\delta_1, \delta_0) \dots$ not isomorphic to it.” in §1.4 has to be replaced by the following:

“We remark that $\text{Ext}^1(\delta_1, \delta_0)$ coincides with $\text{Def}_\nu(\mathbf{k}[\epsilon])$ in the case when $f : X \rightarrow Y$ is a regular embedding. By (11), with $\text{Def}_\nu(\mathbf{k}[\epsilon])$ replaced by $\text{Def}_\nu(\mathbf{k}[\epsilon])/\text{Im}(\Delta)$, one has $\varphi_1 = \lambda - \mu$. Therefore,

$$\text{Ext}_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) \times_{\text{Ext}_{\mathcal{O}_X}^1(\Omega_{Y|X}^1, \mathcal{O}_X)} \text{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1, \mathcal{O}_Y) \simeq \text{Ker}(\varphi_1),$$

Δ coincides with ∂ and Θ with φ_0 . Example 1.7 below gives an instance where $\partial = \Delta$ is nonzero.”

- In the proof of Lemma 2.1, the exact sequence (13) is not exact on the left, but this does not affect the proof.

- Replace the statement of Corollary 2.2 by the following:

Corollary 2.2. There is a natural surjective map

$$\tau : T_{(S,C)}\mathcal{V}_{m,\delta} \longrightarrow \text{Def}_\phi(\mathbf{k}[\epsilon]) \simeq \text{Def}_\nu(\mathbf{k}[\epsilon]).$$

Moreover, if X is stable, then the differential of the moduli map of $\psi_{m,\delta}$ at (S, C) factors as

$$d_{(S,C)}\psi_{m,\delta} : T_{(S,C)}\mathcal{V}_{m,\delta} \xrightarrow{\tau} \text{Def}_\nu(\mathbf{k}[\epsilon]) \longrightarrow \text{Def}_\nu(\mathbf{k}[\epsilon])/\text{Im}(\Delta) \xrightarrow{p_X} \text{Ext}_{\mathcal{O}_X}^1(\Omega_X, \mathcal{O}_X) \simeq T_{[X]}\overline{\mathcal{M}}_g,$$

where p_X is the map appearing in the correct version of (11).

In particular, if $\text{Ext}_{\mathcal{O}_Y}^2(\Omega_Y^1(X), \mathcal{O}_Y) = 0$, then $d_{(S,C)}\psi_{m,\delta}$ is surjective; if

$$\text{Ext}_{\mathcal{O}_Y}^1(\Omega_Y^1(X), \mathcal{O}_Y) = \text{Hom}_{\mathcal{O}_X}(\Omega_Y^1|_X, \mathcal{O}_X) = \text{Hom}_{\mathcal{O}_Y}(\Omega_Y^1, \mathcal{O}_Y) = 0$$

then $d_{(S,C)}\psi_{m,\delta}$ is injective.

- At the end of Remark 2.3, add "In this case, using the above notation, one has $\text{Hom}_{\mathcal{O}_Y}(\Omega_Y^1, \mathcal{O}_Y) = H^0(Y, T_Y) = 0$ and moreover, by [CK, (4) in proof of Prop. 1.2], $\text{Hom}_{\mathcal{O}_X}(\Omega_Y^1|_X, \mathcal{O}_X) = H^0(X, T_{Y|X}) = 0$."

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